Chapter 8

Special Classical Physical Systems

8.1 Introduction

In order to understand the ideas of modern physics, it is essential to understand the operations of some special classical systems. Not only do these provide a physical intuition but also a vocabulary. In the previous chapter, Chapter 7 on page 183, we dealt in some detail with two important physical systems, the free particle and the particle moving in a constant force. These were dealt with there to illustrate the principles and uses of symmetry and action. They obviously belong to the category of "Special Classical Physical Systems" but since they were treated there will not be treated here. Instead we will deal with the harmonic oscillator as an example of a more complicated but still simple system and the string as an example of a field system.

8.2 The Harmonic Oscillator

8.2.1 Importance

After the free particle, the harmonic oscillator is the most important mechanical system. Harmonic oscillators or systems that are almost harmonic oscillators are ubiquitous in nature. These are basically objects that when disturbed slightly return to their starting position but because of inertia overshoot and jiggle back and forth around the neutral position. The simplest example is the simple spring with one end attached to a fixed place with a mass attached on the end.

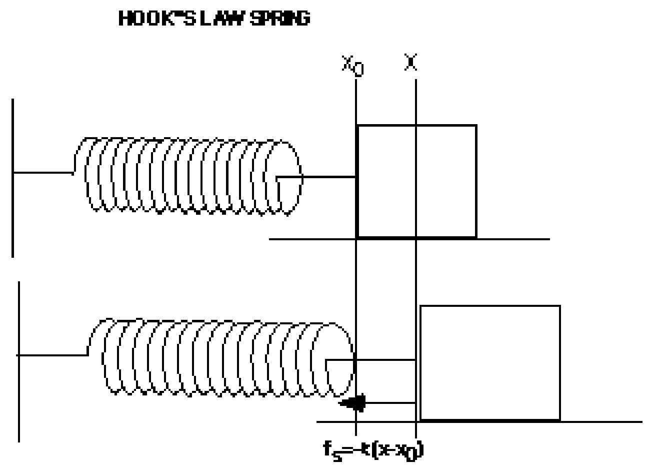

Figure 8.1: **A mass and Hook's Law Spring** A mass, m, on the end of an ideal spring is an example of a harmonic oscillator. An ideal spring or Hook's Law spring, is one in which the force at the end of the spring is proportional to the stretch of the spring, $F = k(x - x_0)$.

The general definition is that the system is a harmonic oscillator if the force on the system that emerges from movement from equilibrium is proportional to the amount of movement from equilibrium and is directed to remove the displacement from equilibrium. Defined this way, harmonic oscillators come in lots of forms. A mass on the end of a string suspended above the earth, if displaced to the side by a small amount is a harmonic oscillator. A shallow pan filled with water sloshes back and forth when disturbed and can be analyzed as a harmonic oscillator. We will discuss these examples in Section 8.2.3 on page 209. In a very real sense, any object that is held in place but still moves a little about that fixed point is generally well approximated by the harmonic oscillator system.

Even more important to our purposes, we will find that the harmonic oscillator is essential to the modern interpretation of the nature of particles. The quantum harmonic oscillator is the only system that can provide a framework for creating a quantum field theory that satisfies the requirements of having a particle interpretation

8.2.2 Dynamics

In the most general case, for a mass that can move freely in space, since acceleration and force are vector quantities $\vec{F} = m\vec{a}$, a harmonic oscillator is a system which obeys:

$$m\vec{a} = -k(\vec{x} - \vec{x_0}), \tag{8.1}$$

where $\vec{a}$ is the acceleration of the position of the block and $\vec{x}$ and $\vec{x_0}$ are the position and neutral position of the mass. k is called the spring constant. The sign is negative since we want the force to drive the system back to the neutral position. What are the dimensions of k? Can you make a time with the dimensional parameters of this problem? Can you make a length? The dimensions of k are $\frac{\text{mass}}{\text{time}^2}$. The only dimensional parameters that involved are k and the mass, m. From these you can make a time, $\sqrt{\frac{m}{k}}$, but you cannot make a length. This lack of an intrinsic length but an intrinsic time will lead to a scaling invariance that is the basis for an interesting property of harmonic oscillators: the period of oscillation is independent of the amplitude of the oscillation.

For most of our purposes, it will be sufficient to deal with only one spatial dimension and, in this section, that is all that will be described. The results in higher spatial dimensions are easily generalized from the one dimensional case.

The Lagrangian for this system is

$$L(v, x) = \frac{m}{2}v^2 - \frac{k}{2}(x - x_0)^2. \tag{8.2}$$

Of course, this Lagrangian yields the correct one dimensional version of the dynamic for this system,

$$ma = -k(x - x_0). \tag{8.3}$$

What are the symmetries and invariances of this system? See Section 7.4 on page 194. Translation in the position coordinate: $x \to x + \alpha$? This is neither a symmetry nor an invariance for this action. Time translation: $t \to t + \alpha$? This is a symmetry and thus there is a conserved quantity, the energy, which we discuss below. A rescale of length: $x \to \lambda x$ produces an invariance. Thus systems with different lengths have the same physics. This is why the period is independent of the amplitude. A rescale of time: $t \to \lambda t$ produces neither a symmetry nor an invariance.

From the Lagrangian, we can construct the energy as the Noether conserved quantity for the time coordinate translation symmetry, see Section 7.3.1 on page 192 and Section 7.4.4 on page 199.

$$E = m\frac{v^2(t)}{2} + k\frac{\{x(t) - x_0\}^2}{2}. \tag{8.4}$$

Identifying the free particle motional energy as $m\frac{v^2}{2}$, there is a potential energy and it is $k\frac{(x-x_0)^2}{2}$. Actually most of you would have done this the other way. The potential energy is $V(x) = k\frac{(x-x_0)^2}{2}$ and the Lagrangian is $K.E. - V(x)$. I just wanted to emphasize the importance of the action approach which is the more fundamental approach.

There are two kinds of motion. If you displace the mass from the equilibrium position, x_0, a distance d, the mass moves as:

$$x(t) = d\cos\left(2\pi\sqrt{\frac{k}{m}}t\right) + x_0 \tag{8.5}$$

It oscillates harmonically about the equilibrium position, x_0, with a radian frequency $\Omega = 2\pi f = 2\pi\sqrt{\frac{k}{m}}$, where f is the usual cycle frequency.

If you have the mass at x_0 and give it an initial velocity, v_0, it moves as:

$$x(t) = \frac{v_0}{2\pi\sqrt{\frac{k}{m}}}\sin\left(2\pi\sqrt{\frac{k}{m}}t\right) + x_0 \tag{8.6}$$

For the general case, you have a superposition of these two motions.

$$x(t) = d\cos\left(2\pi\sqrt{\frac{k}{m}}t\right) + \frac{v_0}{2\pi\sqrt{\frac{k}{m}}}\sin\left(2\pi\sqrt{\frac{k}{m}}t\right) + x_0 \tag{8.7}$$

The velocity is

$$v(t) = -d2\pi\sqrt{\frac{k}{m}}\sin\left(2\pi\sqrt{\frac{k}{m}}t\right) + v_0\cos\left(2\pi\sqrt{\frac{k}{m}}t\right) \tag{8.8}$$

This provides a wonderful example of a conserved quantity. Both $x(t)$ and $v(t)$ are changing all the time. Even the kinetic energy is changing and the potential energy is changing. Only when you take the combination of $E(t) = K.E.(t) + P.E.(t)$ do you get something that does not change. Plug

Equation 8.7 on page 208 for $x(t)$ and Equation 8.8 on page 208 for $v(t)$ into Equation 8.4 on page 208 and get that

$$\begin{aligned} E &= m\frac{(v(t))^2}{2} + k\frac{(x(t)-x_0)^2}{2} \\ &= m\frac{v_0^2}{2} + k\frac{d^2}{2}. \end{aligned} \tag{8.9}$$

8.2.3 Examples of harmonic oscillator systems

Besides being a nice simply solvable example of dynamical system, the oscillator is a very common example. Almost all systems that undergo bounded motion, act like an oscillator for small ranges of motion.

Consider the pendulum, a mass on the end of a flexible string suspended freely above the earth. This is certainly a case of bounded motion. How is it related to the harmonic oscillator system?

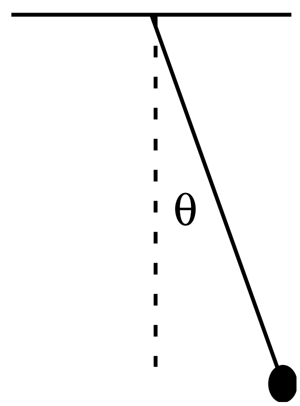

Figure 8.2: **Simple Pendulum** Simple Pendulum.

As is always the case in classical physics, the Lagrangian is $K.E. - P.E.$. For this case, the $K.E.$ is the usual $m\frac{v^2}{2}$. The $P.E.$ is our old friend mgh but, in this case, we want the dynamical variable to be the angle of the string from the vertical, θ. Using $h = l(1-\cos(\theta))$, we have for the Lagrangian:

$$L(v,\theta) = m\frac{v^2}{2} - mgl(1-\cos(\theta)) \tag{8.10}$$

where l is the length of the string in the pendulum. Again, there is a time translation symmetry and, with the use of Noether's Theorem, Sections 7.3.1 on page 192 and 7.4.4 on page 199, we can construct a conserved quantity called the energy and we can identify the kinetic and potential energies. In this case, the potential energy is $V(\theta) = mgl(1 - \cos(\theta))$. Using the information from Section 1.4.2 on page 16, "Things Everyone Should Know" for small θ, $V(\theta) \simeq mgl\frac{\theta^2}{2}$. Also the kinetic energy is not directly related to how fast the angle θ is changing. Since this is our dynamical variable, we want to express the $K.E.$ in terms of the rate of change. The linear speed, v is connected to the angular speed, $\frac{\Delta\theta}{\Delta t} \equiv \omega$ as $v = l\omega$. For small angles the pendulum has as its Lagrangian:

$$L(\omega, \theta) = ml^2\frac{\omega^2}{2} - mgl\frac{\theta^2}{2}. \tag{8.11}$$

Making a correspondence between v and ω and x and θ, we see that, in the limit of small θ and comparing to Equation 8.2 on page 207, the pendulum is an example of a harmonic oscillator. In other words, if we consider ml^2 to be an effective mass and mgl to be an effective spring constant the pendulum moves in exactly the same way as the harmonic oscillator. This means that the motion is harmonic about the equilibrium position, $\theta_0 = 0$, with radian frequency $\Omega = 2\pi\sqrt{\frac{mgl}{ml^2}} = 2\pi\sqrt{\frac{g}{l}}$. If you have a starting displacement θ_d and starting angular speed ω_0,

$$\theta(t) = \theta_d \cos\left(2\pi\sqrt{\frac{g}{l}}t\right) + \frac{\omega_0}{2\pi\sqrt{\frac{g}{l}}} \sin\left(2\pi\sqrt{\frac{g}{l}}t\right). \tag{8.12}$$

8.3 The Stretched String Revisited

8.3.1 Distributed Systems

Instead of having any masses concentrated into point, mass can be distributed. An example is the stretched elastic string, see Section 5.3 on page 128.

This is an example of a field. The disturbance of the string is defined at every point on the stretched string and the disturbance will have a dynamic, a rule for its evolution.

Let me review that physics of the string between fixed walls. The electromagnetic field has many of the same properties. It is just a more complex field and the complications do not add any to the understanding of the quantum properties of the field. The stretched string is a one dimensional field where that field variable, $y(x, t)$, is the transverse displacement of the string from its equilibrium position and x is the distance along the interval between the walls. The electromagnetic field is three dimensional.

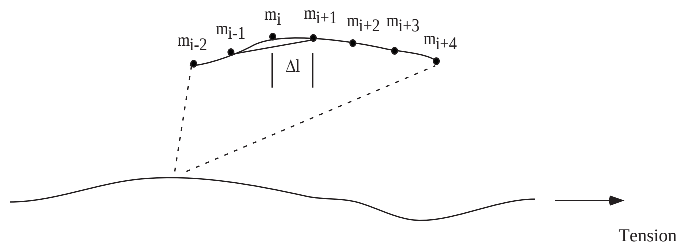

Figure 8.3: **A stretched string** The stretched string can be envisioned as a sequence of masses separated by a distance Δl which are connected by elastic elements which provide verticle force on the mass elements.

The dynamics of the string are well understood. The rule is very simple. The net vertical force on a piece of string of length Δl which equals the mass of that length of string times the acceleration of the transverse displacement is proportional to the negative of the displacement from the average of the displacements of its neighbors. The proportionality constant has the dimensions of a force per unit length and is thus the tension in the string divided by the length of the piece of string. ρ is the mass per unit length of the string.

$$\rho \Delta l a_{x,t} = -\frac{T}{\Delta l}\left[y(x,t) - \frac{\left(y(x+\frac{\Delta l}{2},t) + y(x-\frac{\Delta l}{2},t)\right)}{2}\right] \tag{8.13}$$

You can also derive this result by cutting the string and seeing how the tension acts to stretch the string.

In the limit that Δl is zero this goes to

$$\rho\frac{\partial^2 y}{\partial t^2}(x,t) = T\frac{\partial^2 y}{\partial x^2}(x,t) \tag{8.14}$$

Note that $\frac{T}{\rho} \stackrel{\text{dim}}{=} \frac{\frac{ML}{T^2}}{\frac{M}{L}} \stackrel{\text{dim}}{=} \frac{L^2}{T^2}$ has the dimensions of a velocity squared. From this we can intuit that the disturbances in the field travel with a velocity $v = \pm\sqrt{\frac{T}{\rho}}$.

Any configuration of the displacements of a stretched string is a superposition of normal modes. When you pluck a stretched string you generally put in a localized disturbance. This excites all the modes and the higher frequency modes will damp out quickly and you are left with the fundamental.

Using the normal modes the stretched string can be considered a countable infinity of oscillators.

The quantum particle that is at its basis is called the phonon.

The photon is a state of the electromagnetic field that has a definite frequency, $\hbar\omega$. This implies that the field configuration is a normal mode. In other words, there is a photon foreach of the normal modes. To understand the implications of this statement consider the stretched string.

8.3.2 Concluding Remarks

At the end of the 19^{th} century, we had a unified physics using these action principles for particles and fields and their interactions. The name of the game was to write down the Lagrangian for the particle motions and the fields. Do the least action machinery and you knew all the conserved quantities and what was happening. There were only two fundamental forces, electromagnetism and gravitation. Both were well described by action principles, one a field theory and the other an action at a distance theory. All higher order phenomena were felt to be described by these fundamental entities. There was a feeling expressed by some that we may be near to the end of physics. This was clearly naive. Even on the face of it, there were clear problems that would require new insights. Why was the basis of physics built on such different mechanisms– field theory and action at a distance? What was the underlying machinery that could unify this physics? Despite the theoretical questions, the real basis for discovering a new physics would be the new experimental developments that took place at the turn of the century.